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NONLINEAR VIBRATION MODES OF AN ELASTIC PANEL UNDER PERIODIC LOADING

A. L. Tukmakov

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The dynamic instability and nonlinear behavior of a nonshallow thin elastic cylindrical panel with simply supported rectilinear edges under uniformly distributed periodic load is studied. The regions of regular and chaotic dynamics are determined for symmetric and nonsymmetric bending modes of the panel. It is shown that depending on the external load frequency, the nonsymmetric buckling. which occurs when the load amplitude reaches a critical value, can lead to two different dynamic modes.

The motion of a nonlinear determinate system can be regular or chaotic, depending on the values of the control parameters [1, 2]. By the regular motion, the periodic or quasiperiodic mode is meant. If, however, the system is very sensitive to initial conditions, for certain values of the control parameters, the chaotic mode of motion can occur in which the energy is intensively transferred to the low-frequency region. The motion of an elastic panel whose deflections are greater than or comparable with the panel thickness is described by a geometrically nonlinear system of equations; therefore, chaotic modes can also be realized [3–5] in this system.

The motion of the panel is described by the following system of geometrically nonlinear equations of the shell theory based on the Kirchhoff–Love hypothesis [6, 7]:

$$\begin{gathered}\frac{1}{T^2}\frac{\partial^2 V}{\partial t^2} - (1+\varepsilon)\frac{\partial^2 V}{\partial \alpha^2} + \frac{\partial W}{\partial \alpha} - \varepsilon\frac{\partial^3 W}{\partial \alpha^3} + F_1(V,W) = Z_\tau,\\ \frac{1}{T^2}\frac{\partial^2 W}{\partial t^2} + \gamma\frac{\partial W}{\partial t} + \varepsilon\frac{\partial^4 W}{\partial \alpha^4} + \varepsilon\frac{\partial^3 V}{\partial \alpha^3} - \frac{\partial V}{\partial \alpha} + F_2(V,W) = Z_n.\end{gathered} \tag{1}$$

Here Z_τ and Z_n are the tangential and normal components of the dynamic load and F_1 and F_2 are the terms containing the following nonlinear terms:

$$F_1 = -\frac{\partial W}{\partial \alpha} + \Big(\frac{\partial V}{\partial \alpha} - W\Big)\Big(\frac{\partial^2 V}{\partial \alpha^2} - \frac{\partial W}{\partial \alpha}\Big) + \Big(\frac{\partial W}{\partial \alpha} + V\Big)\Big(\frac{\partial^2 W}{\partial \alpha^2} + \frac{\partial W}{\partial \alpha}\Big) + \varepsilon\Big(\Big(\frac{\partial W}{\partial \alpha} + V\Big)\Big(\frac{\partial^2 W}{\partial \alpha^2} - \frac{\partial^3 V}{\partial \alpha^3}\Big)$$

$$+ \frac{\partial^3 W}{\partial \alpha^3} + \Big(\frac{\partial V}{\partial \alpha} - W\Big)\Big(\frac{\partial^3 W}{\partial \alpha^3} + \frac{\partial^2 V}{\partial \alpha^2}\Big)\Big) + \varepsilon\Big(\Big(\frac{\partial^2 W}{\partial \alpha^2} + V\Big)\Big(\frac{\partial^2 W}{\partial \alpha^2} - \frac{\partial^3 V}{\partial \alpha^3}\Big)$$

$$+ \Big(1 + \frac{\partial V}{\partial \alpha} - W\Big)\Big(\frac{\partial^3 W}{\partial \alpha^3} + \frac{\partial^2 V}{\partial \alpha^2}\Big)\Big)\Big(\Big(\frac{\partial W}{\partial \alpha} + V\Big)\Big(\frac{\partial W}{\partial \alpha} - \frac{\partial^2 V}{\partial \alpha^2}\Big) + \Big(1 + \frac{\partial V}{\partial \alpha} - W\Big)\Big(\frac{\partial^2 W}{\partial \alpha^2} + \frac{\partial V}{\partial \alpha}\Big)\Big),$$

$$F_2 = -\varepsilon\frac{\partial^3 V}{\partial \alpha^3} + \frac{\partial V}{\partial \alpha} + \frac{1}{2}\Big(\frac{\partial V}{\partial \alpha} - W\Big)^2 + \frac{1}{2}\Big(\frac{\partial W}{\partial \alpha} + V\Big)^2 + \varepsilon\Big(\Big(\frac{\partial^2 W}{\partial \alpha^2} + \frac{\partial V}{\partial \alpha}\Big)\Big(\frac{\partial^3 V}{\partial \alpha^3} - \frac{\partial^2 W}{\partial \alpha^2}\Big)$$

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$$+\Big(\frac{\partial W}{\partial\alpha}+V\Big)\Big(\frac{\partial^4 V}{\partial\alpha^4}-\frac{\partial^3 W}{\partial\alpha^3}\Big)-\Big(\frac{\partial^2 V}{\partial\alpha^2}-\frac{\partial W}{\partial\alpha}\Big)\Big(\frac{\partial^3 W}{\partial\alpha^3}+\frac{\partial^2 V}{\partial\alpha^2}\Big)-\Big(\frac{\partial V}{\partial\alpha}-W\Big)\Big(\frac{\partial^4 W}{\partial\alpha^4}+\frac{\partial^3 V}{\partial\alpha^3}\Big)\Big)$$
$$+\Big(\frac{\partial V}{\partial\alpha}-W+\frac{1}{2}\Big(\frac{\partial V}{\partial\alpha}-W\Big)^2+\frac{1}{2}\Big(\frac{\partial W}{\partial\alpha}+V\Big)^2\Big)\Big(\Big(\frac{\partial W}{\partial\alpha}+V\Big)\Big(\frac{\partial W}{\partial\alpha}-\frac{\partial^2 V}{\partial\alpha^2}\Big)$$
$$+\Big(1+\frac{\partial V}{\partial\alpha}-W\Big)\Big(\frac{\partial^2 W}{\partial\alpha^2}+\frac{\partial V}{\partial\alpha}\Big)\Big);$$

where W and V are the deflection and tangential displacement of fixed (Lagrangian) points of the middle surface which are normalized to the radius of the undeformed panel R, $\varepsilon=\delta^2/12$, $\delta=h/R$, $T^2=E/(\rho(1-\mu^2))$, h and ρ are the thickness and density of the panel material, respectively, E is Young's modulus, and μ is Poisson's ratio. The term $\gamma(\partial W/\partial t)$, which accounts for structural damping, is introduced into the deflection equation. The deflection is positive if directed toward the curvature center, and the tangential displacement is positive in the counterclockwise direction. The simply supported boundary conditions at the rectilinear edges of the panel are written in the form

$$W=0,\quad V=0,\quad \frac{\partial^2 W}{\partial\alpha^2}=0 \quad \text{for} \quad \alpha=\alpha_1,\quad \alpha=\alpha_2. \tag{2}$$

It is assumed that the panel is immovable at the initial moment of time:

$$W=0,\quad V=0,\quad \frac{\partial W}{\partial t}=0,\quad \frac{\partial V}{\partial t}=0 \quad \text{for} \quad t=0,\quad \alpha_1\leqslant\alpha\leqslant\alpha_2. \tag{3}$$

The normal component of dynamic load Z_n changes periodically in time, whereas the tangential component Z_τ is equal to zero:

$$Z_n=AD\sin(\omega t),\qquad Z_\tau=0. \tag{4}$$

Here A and ω are the amplitude and frequency of the external load, respectively, and $D=Eh^3/(12(1-\mu^2))$ is the flexural rigidity of the panel. The load is assumed to be rigid and independent of the middle-surface shape.

Subject to conditions (2)–(4), system (1) was solved by the finite-difference method with the use of second-order implicit difference schemes [7–9]. For analysis of the solution, the signal-power spectrum

$$|\bar X_k|^2=\Big|\frac{1}{\sqrt n}\sum_{j=1}^{n}X_j\exp\Big(-i\,\frac{2\pi kj}{n}\Big)\Big|^2$$

was constructed, where $|\bar X_k|^2$ is the discrete component of the power spectrum, which depends on the frequency, X_j is the deflection at the central point of the panel at the moment $t_j=j\Delta t$, and Δt is the time step.

In accordance with the Wiener–Khintchine theorem (with accuracy up to a numerical factor), the autocorrelative function ψ_k was defined as the Fourier transform of the power spectrum at the moment $t_k=k\Delta t$ [1, 2]:

$$\psi_k=\frac{1}{\sqrt n}\sum_{j=1}^{n}|\bar X_j|^2\exp\Big(-i\,\frac{2\pi kj}{n}\Big).$$

To analyze the evolution of the signal and its chaotic state, we determined the dynamic and static characteristics of the attractor: the Shannon information and the lower bound of the Hausdorff dimensionality D_2 [1, 2]. To introduce a set of states into the phase space (W, dW/dt, d^2W/dt^2), the region occupied by the attractor was covered by a grid with the cell size l. Belonging of an attractor point to a fixed cell of the grid is understood to be the state of the system. As the dynamic characteristic, we used the average value of the increment in the Shannon information [2]

$$\Delta\bar I=\frac{1}{N}\sum_{n=1}^{N}\Delta I_n,$$

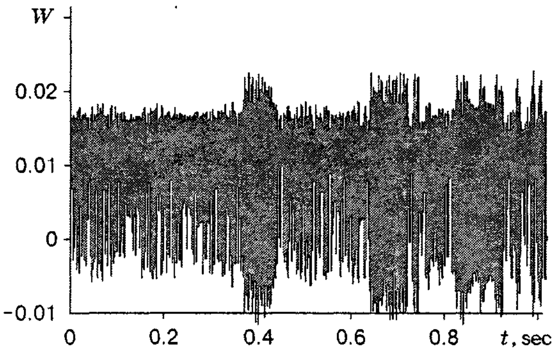

Fig. 1

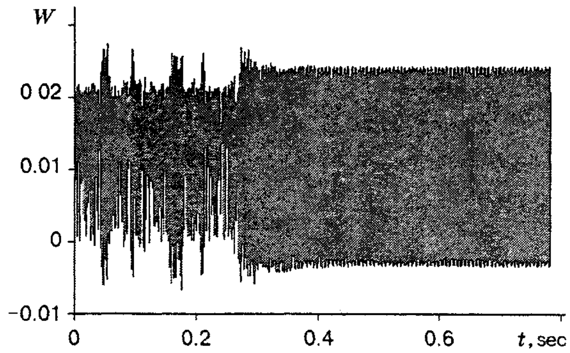

Fig. 2

where $\Delta I_n = I_{n+1} - I_n$, $I_n = \sum_{i=1}^{J_1} P_i \log_2 P_i$, and $I_{n+1} = \sum_{i=1}^{J_2} P_i \log_2 P_i$.

Based on the analysis of the signal, at each moment of time t_j we determined the state of the system n and the sets of states M_1 and M_2 to which the system can change at subsequent moments of time t_{j+1} and t_{j+2}. The sets M_1 and M_2 contain J_1 and J_2 different states, respectively. In the expressions for the information I_n and I_{n+1}, the probabilities of transition of the system from the state n to certain states from the sets M_1 and M_2, respectively, are denoted by P_i. If the average value of the increment in the Shannon information is positive, the average number of states to which the system can change at a given moment of time is greater than unity and, hence, the dynamic process is chaotic. If this quantity is equal to zero, the transition from any state to a subsequent state is unique and the dynamics is regular [2]. The values of J_1, J_2, and P_i were determined by analyzing the attractor in the space of states of the phase system.

As a static characteristic of the attractor, the lower bound of the Hausdorff dimensionality determined by means of correlation integral [2] was used:

$$D_2 = \lim_{l \to 0} (\ln C(l) / \ln l), \qquad C(l) = \lim_{l \to 0} (1/N^2) \sum_{i,j} \theta(l - |\boldsymbol{x}_i - \boldsymbol{x}_j|).$$

Here l is the edge length of the cubic cells into which the region occupied by the attractor is divided, N is the number of points of the attractor, θ is the Heaviside function, and $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are the radius-vectors of the attractor points in the phase space. The quantity D_2 was determined as the slope of the linear section (outside the saturation region and the region where statistic information is insufficient) of the curve $\log_2 l$ versus $\log_2 C(l)$ to the $\log_2 l$ axis.

In accordance with the technique described above, the dynamic behavior of the panel was investigated as a function of the amplitude value of the external load at a given frequency. Calculations were performed for the following parameters of the panels: $\delta = 0.01$, $\rho = 4500$ kg/m^3, $T = 5000$ m/sec, and $\gamma = 0.0001$.

Nonlinear Symmetric Vibrations. We consider the calculation results obtained for a panel with the curvature parameter $K = 4L^2/(Rh) = 4$ ($2L$ is the panel span) at which the bending is symmetric [10]. The external load varies according to the harmonic law (4) with a frequency equal to the first natural frequency of the elastic panel. Figure 1 shows the deflection at the central point of the panel W versus time t for the minimum amplitude of the external load $A = 150$ that ensures vibrations in two states of equilibrium. An attractor with the lower bound of the Hausdorff dimensionality $D_2 = 2.15$ forms in the phase space. Vibrations occur relative to the two stable states of equilibrium. The average increment in the Shannon information is positive $\Delta \bar{I} = 0.7236$; the power spectrum contains a continuous low-frequency component; the autocorrelative function decreases on the interval of its definition. According to the criteria considered [1, 2], the vibrations are chaotic for these panel and load parameters.

We now increase the amplitude of the external load up to $A = 200$ with the panel parameters and the

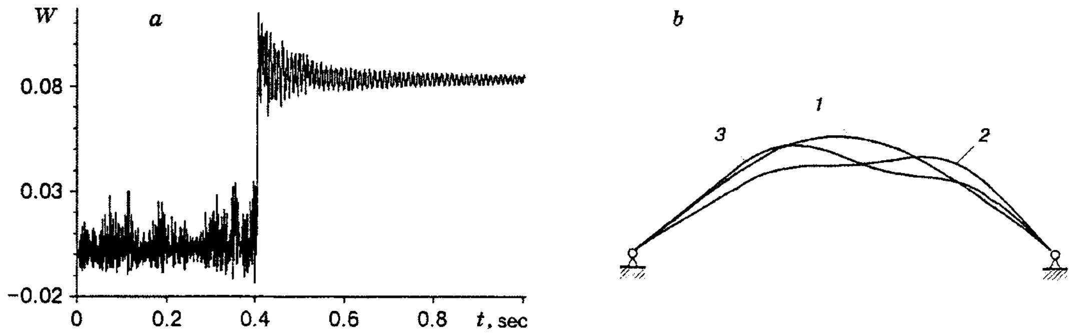

Fig. 3

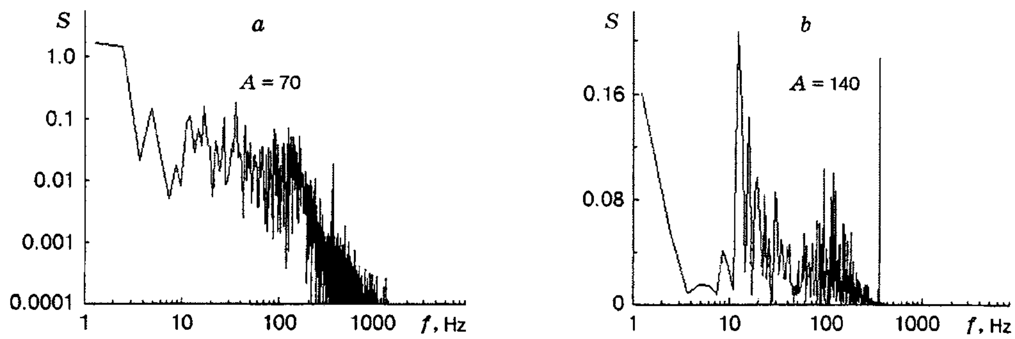

Fig. 4

boundary conditions preserved. Figure 2 shows the deflection W at the central point of the panel versus the time t. Chaotic splashes of the deflection amplitude are observed at the initial stage of vibrations. Beginning with $t = 0.4$ sec, the vibration mode tends to a quasiperiodic mode. This follows from the decrease in the power of the continuous low-frequency component relative to the power of the external load and the character of the autocorrelative function, which oscillates about its slowly changing mean value. For $t > 0.4$ sec, the attractor approaches the limit cycle and its dimensionality decreases to $D_2 = 1.21$, and the average increment in the Shannon information is $\Delta\bar{I} = 0$. An analysis of the Poincare section of the attractor by the plane $W = 0.5$ in the three-dimensional phase space shows that the attractor is compressed in this plane. The results show that, for the chosen parameters of the problem, the chaotic mode changes to a quasiperiodic mode provided the amplitude of the load is sufficiently large. If the amplitude of vibrations is small and does not exceed the panel thickness, quasiperiodic vibrations occur in the system. Thus, if the amplitude of the external load is used as a control parameter, the region of chaotic dynamics lies between two regions of regular dynamics.

Nonlinear Nonsymmetric Vibrations. For the curvature parameter $K = 4L^2/(Rh) > 9.04$ [10], the elastic panel executes nonsymmetric vibrations. As an example, we consider the calculation results for $K = 19.6$, $\delta = 0.01$, $\rho = 4500$ kg/m^3, $T = 5000$ m/sec, and $\gamma = 0.0001$. In this case, the first natural frequency of the undeformed panel is $f_1 = 500$ Hz. Figure 3 shows the deflection at the central point of the panel W versus time (a) and the shapes of the panel for different times (b) (curves 1–3 refer to $t = 0.01$, 0.02, and 0.03 sec, respectively). These results were obtained for a frequency of the external load equal to the first resonance frequency and the minimum amplitude $A = 60$ at which the panel snaps. The characteristic feature of this mode is that the panel does not snap back. This is due to the fact that the resonance frequencies for the two states of equilibrium differ significantly. Namely, for the initial position, the first natural vibration frequency is $f_1 = 500$ Hz, whereas $f_1 = 100$ Hz in the snapped position. Since the panel loses its stability at the resonance frequency, the amplitude of the load is not very large. In the snapped state, the resonance frequency does not coincide with the load frequency, which results in regular quasiperiodic vibrations about the lower stable position (Fig. 3b).

As the amplitude of the load gradually increases at a frequency shifted from the resonance frequency, the panel snaps and then executes irregular vibrations, reaching the two positions of equilibrium. Figure 4a shows the power of vibrations at the central point of the panel S versus the vibration frequency f for the external-load frequency $f_{\text{load}} = 350$ Hz and the minimum amplitude $A = 70$ at which the instability occurs. In this case, the amplitude of the critical load is larger and the power supplied is sufficient to provide vibrations with the maximum possible amplitude of deflection. The power spectrum of vibrations contains a continuous component in the low-frequency region; therefore, the dynamic mode can be classified as a chaotic mode. If the load amplitude increases ($A = 140$), the low-frequency component of the power spectrum decreases relative to the frequency of the external load (Fig. 4b). Nevertheless, in view of its presence in the power spectrum, the dynamic mode cannot be classified as a quasiperiodic mode.

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